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On Explicit Point Multi-Monopoles in SU(2) Gauge Theory

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Abstract

It is well known that the Dirac monopole solution with the U(1) gauge group embedded into the group SU(2) is equivalent to the SU(2) Wu-Yang point monopole solution having no Dirac string singularity. We consider a multi-center configuration of m Dirac monopoles and n anti-monopoles and its embedding into SU(2) gauge theory. Using geometric methods, we construct an explicit solution of the SU(2) Yang-Mills equations which generalizes the Wu-Yang solution to the case of m monopoles and n anti-monopoles located at arbitrary points in $\mathbb{R}^3$.

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1 Introduction

Abelian magnetic monopoles play a key role in the dual superconductor mechanism of confinement [1] which has been confirmed by many numerical simulations of the lattice gluodynamics (see e.g. [2, 3] and references therein). Due to a dominant role of abelian monopoles in the confinement phenomena, it is important to understand better how do they arise in nonabelian pure gauge theories.

A spherically-symmetric monopole solution of the SU(2) pure gauge field equations was obtained by Wu and Yang in 1969 [4]. This solution is singular at the origin and smooth on $\mathbb{R}^3-\{0\}$. Initially it was thought that it is genuinely nonabelian, yet later it was shown [5] that this solution is nothing but the abelian Dirac monopole [6] in disguise. Note that the gauge potential of the finite-energy spherically symmetric 't Hooft-Polyakov monopole [7] approaches just the Wu-Yang gauge potential for large $r^2 = x^a x^a$.

In this note, we generalize the Wu-Yang solution to a configuration describing m monopoles and n anti-monopoles with arbitrary locations in $\mathbb{R}^3$. This explicit solution to the Yang-Mills equations can also be used as a guide to the asymptotic $r \to \infty$ behaviour of unknown finite-energy solutions in Yang-Mills-Higgs theory, whose form for small r is determined by multiplying the solution by arbitrary functions and minimizing the energy functional, as was proposed in [8].

2 Generic U(1) configurations

We consider the configuration of m Dirac monopoles and n anti-monopoles located at points $\vec{a}_i = \{a_i^1, a_i^2, a_i^3\}$ with $i = 1, \dots, m$ and $i = m+1, \dots, m+n$, respectively. There are delta-function sources for the magnetic field at these points.

Let us introduce the following two regions in $\mathbb{R}^3$:

$$R^3_{N,m+n} := \mathbb{R}^3 - \left\{ \bigcup_{i=1}^{m+n} (x^1 = a_i^1,\, x^2 = a_i^2,\, x^3 \le a_i^3) \right\} \ ,$$

$$R^3_{S,m+n} := \mathbb{R}^3 - \left\{ \bigcup_{i=1}^{m+n} (x^1 = a_i^1,\, x^2 = a_i^2,\, x^3 \ge a_i^3) \right\} \ . \tag{1}$$

For simplicity we restrict ourselves to the *generic* case

$$a_i^{1,2} \ne a_j^{1,2} \quad \text{for} \quad i \ne j \ , \tag{2}$$

when

$$R^3_{N,m+n} \ \cup \ R^3_{S,m+n} = \mathbb{R}^3 - \{\vec{a}_1, \dots, \vec{a}_{m+n}\} \ , \tag{3}$$

and the two open sets are enough for describing the above (m,n)-configuration. Namely, the generic configuration of m Dirac monopoles and n anti-monopoles is described by the gauge potentials

$$A^{N,m+n} = \sum_{j=1}^{m} A^{N,j} + \sum_{j=m+1}^{m+n} \bar{A}^{N,j} \quad \text{and} \quad A^{S,m+n} = \sum_{j=1}^{m} A^{S,j} + \sum_{j=m+1}^{m+n} \bar{A}^{S,j} \ , \tag{4}$$

where $A^{N,m+n}$ and $A^{S,m+n}$ are well defined on $R^3_{N,m+n}$ and $R^3_{S,m+n}$, respectively. Here

$$A^{N,j} = A_a^{N,j}\mathrm{d}x^a \quad \text{with} \quad A_1^{N,j} = \frac{\mathrm{i}x_j^2}{2r_j(r_j + x_j^3)} \ , \quad A_2^{N,j} = -\frac{\mathrm{i}x_j^1}{2r_j(r_j + x_j^3)} \ , \quad A_3^{N,j} = 0 \ , \tag{5}$$

$$A^{S,j} = A^{S,j}_a \mathrm{d}x^a \quad \text{with} \quad A^{S,j}_1 = -\frac{\mathrm{i}x^2_j}{2r_j(r_j - x^3_j)} \ , \quad A^{S,j}_2 = \frac{\mathrm{i}x^1_j}{2r_j(r_j - x^3_j)} \ , \quad A^{S,j}_3 = 0 \ , \tag{6}$$

where

$$x^c_j = x^c - a^c_j \ , \quad r^2_j = \delta_{ab} x^a_j x^b_j \ , \quad a, b, c = 1, 2, 3 \ , \tag{7}$$

and $\bar{A}^{N,j} = -A^{N,j}$, $\bar{A}^{S,j} = -A^{S,j}$. On the intersection $R^3_{N,m+n} \cap R^3_{S,m+n}$ we have

$$A^{N,m+n} = A^{S,m+n} + \mathrm{d} \, \ln \left(\prod_{i=1}^{m} \left(\frac{\bar{y}_i}{y_i} \right)^{\frac{1}{2}} \prod_{j=m+1}^{m+n} \left(\frac{y_j}{\bar{y}_j} \right)^{\frac{1}{2}} \right) \ , \tag{8}$$

where $y_j = x^1_j + \mathrm{i}x^2_j$ and bar denotes a complex conjugation.

Remark. Note that in the case when $a^{1,2}_i = a^{1,2}_j$ for some $i \neq j$, one has to introduce more than two open sets covering the space $\mathbb{R}^3 - \{\vec{a}_1, \ldots, \vec{a}_{m+n}\}$ and define gauge potentials on each of these sets as well as transition functions on their intersections. However, for the case $\vec{a}_1 = \ldots = \vec{a}_{m+n} = \vec{a}$ the two sets (1) are again enough to cover $\mathbb{R}^3 - \{\vec{a}\}$ and the gauge potential (4)-(6) will describe $m - n$ monopoles (if $m > n$) or $n - m$ anti-monopoles (if $m < n$) sitting on top of each other.

One can simplify expressions (4)-(8) by introducing functions of coordinates

$$w_j := \frac{y_j}{r_j - x^3_j} = \mathrm{e}^{\mathrm{i}\varphi_j} \cot \frac{\vartheta_j}{2} \quad \text{and} \quad v_j := \frac{1}{w_j} = \frac{\bar{y}_j}{r_j + x^3_j} = \mathrm{e}^{-\mathrm{i}\varphi_j} \tan \frac{\vartheta_j}{2} \ , \tag{9}$$

where

$$x^1_j = r_j \, \sin \vartheta_j \, \cos \varphi_j \ , \quad x^2_j = r_j \, \sin \vartheta_j \, \sin \varphi_j \quad \text{and} \quad x^3_j = r_j \, \cos \vartheta_j \ . \tag{10}$$

Note that $w_i \to \infty$ for $x^{1,2} \to a^{1,2}_i$, $x^3 \geq a^3_i$, and $v_i \to \infty$ for $x^{1,2} \to a^{1,2}_i$, $x^3 \leq a^3_i$. In terms of w_j and v_j the gauge potentials (4)-(6) have the form

$$A^{N,m+n} = \sum_{i=1}^{m} \frac{1}{2(1 + v_i \bar{v}_i)} (\bar{v}_i \mathrm{d}v_i - v_i \mathrm{d}\bar{v}_i) + \sum_{i=m+1}^{m+n} \frac{1}{2(1 + v_i \bar{v}_i)} (v_i \mathrm{d}\bar{v}_i - \bar{v}_i \mathrm{d}v_i) \ , \tag{11}$$

$$A^{S,m+n} = \sum_{i=1}^{m} \frac{1}{2(1 + w_i \bar{w}_i)} (\bar{w}_i \mathrm{d}w_i - w_i \mathrm{d}\bar{w}_i) + \sum_{i=m+1}^{m+n} \frac{1}{2(1 + w_i \bar{w}_i)} (w_i \mathrm{d}\bar{w}_i - \bar{w}_i \mathrm{d}w_i) \ . \tag{12}$$

On the intersection $R^3_{N,m+n} \cap R^3_{S,m+n}$ of two domains (1) these configurations are related by the transformation

$$A^{N,m+n} = A^{S,m+n} + \mathrm{d} \, \ln \left(\prod_{i=1}^{m} \left(\frac{\bar{w}_i}{w_i} \right)^{\frac{1}{2}} \prod_{j=m+1}^{m+n} \left(\frac{w_j}{\bar{w}_j} \right)^{\frac{1}{2}} \right) , \tag{13}$$

since $\bar{y}_i / y_i = \bar{w}_i / w_i$. Note that the transition function in (13) can also be written in terms of v_i by using the relation $v_i / \bar{v}_i = \bar{w}_i / w_i$.

For the abelian curvature $F^{D,m+n}$ we have

$$F^{D,m+n} = \mathrm{d} \, A^{N,m+n} = -\sum_{i=1}^{m} \frac{\mathrm{d}v_i \wedge \mathrm{d}\bar{v}_i}{(1 + v_i \bar{v}_i)^2} + \sum_{i=m+1}^{m+n} \frac{\mathrm{d}v_i \wedge \mathrm{d}\bar{v}_i}{(1 + v_i \bar{v}_i)^2} =$$

$$= -\sum_{i=1}^{m} \frac{\mathrm{d}w_i \wedge \mathrm{d}\bar{w}_i}{(1+w_i\bar{w}_i)^2} + \sum_{i=m+1}^{m+n} \frac{\mathrm{d}w_i \wedge \mathrm{d}\bar{w}_i}{(1+w_i\bar{w}_i)^2} = \mathrm{d}\, A^{S,m+n} \ . \tag{14}$$

It is not difficult to see that $F^{D,m+n}$ is singular only at points $\{\vec{a}_1, \dots, \vec{a}_{m+n}\}$, where monopoles and anti-monopoles are located.

3 Point SU(2) configurations

The generalization of the Wu-Yang SU(2) monopole [4] to a configuration describing m monopoles and n anti-monopoles can be obtained as follows. Let us multiply equation (13) by the Pauli matrix σ_3 and rewrite it as

$$A^{N,m+n}\sigma_3 = f_{NS}^{(m,n)} A^{S,m+n}\sigma_3 \big(f_{NS}^{(m,n)}\big)^{-1} + f_{NS}^{(m,n)} \mathrm{d}\big(f_{NS}^{(m,n)}\big)^{-1} \ , \tag{15}$$

where

$$f_{NS}^{(m,n)} = \begin{pmatrix} \prod\limits_{i=1}^{m} \left(\frac{w_i}{\bar{w}_i}\right)^{\frac{1}{2}} \prod\limits_{j=m+1}^{m+n} \left(\frac{\bar{w}_j}{w_j}\right)^{\frac{1}{2}} & 0 \\ 0 & \prod\limits_{i=1}^{m} \left(\frac{\bar{w}_i}{w_i}\right)^{\frac{1}{2}} \prod\limits_{j=m+1}^{m+n} \left(\frac{w_j}{\bar{w}_j}\right)^{\frac{1}{2}} \end{pmatrix} . \tag{16}$$

It can be checked by direct calculation that the transition matrix (16) can be splitted as

$$f_{NS}^{(m,n)} = (g_N^{(m,n)})^{-1} g_S^{(m,n)} \ , \tag{17}$$

where the 2×2 unitary matrices

$$g_N^{(m,n)} = \frac{1}{\big(1 + \prod\limits_{i=1}^{m+n} v_i\bar{v}_i\big)^{\frac{1}{2}}} \begin{pmatrix} \prod\limits_{j=1}^{m} v_j \prod\limits_{k=m+1}^{m+n} \bar{v}_k & 1 \\ -1 & \prod\limits_{j=1}^{m} \bar{v}_j \prod\limits_{k=m+1}^{m+n} v_k \end{pmatrix} \tag{18}$$

and

$$g_S^{(m,n)} = \frac{1}{\big(1 + \prod\limits_{i=1}^{m+n} w_i\bar{w}_i\big)^{\frac{1}{2}}} \begin{pmatrix} 1 & \prod\limits_{j=1}^{m} \bar{w}_j \prod\limits_{k=m+1}^{m+n} w_k \\ -\prod\limits_{j=1}^{m} w_j \prod\limits_{k=m+1}^{m+n} \bar{w}_k & 1 \end{pmatrix} \tag{19}$$

are well defined on $R^3_{N,m+n}$ and $R^3_{S,m+n}$, respectively. Using formulae (9) and (10), one can rewrite these matrices in the coordinates x_i^a with explicit dependence on moduli $\vec{a}_i$ for $i = 1, \dots, m+n$.

Substituting (17) into (15), we obtain

$$A^{N,m+n} g_N^{(m,n)} \sigma_3 \big(g_N^{(m,n)}\big)^{\dagger} + g_N^{(m,n)} \mathrm{d}\big(g_N^{(m,n)}\big)^{\dagger} = A^{S,m+n} g_S^{(m,n)} \sigma_3 \big(g_S^{(m,n)}\big)^{\dagger} + g_S^{(m,n)} \mathrm{d}\big(g_S^{(m,n)}\big)^{\dagger} =: A_{su(2)}^{(m,n)} , \tag{20}$$

where by construction $A_{su(2)}^{(m,n)}$ is well defined on $R^3_{N,m+n} \cup R^3_{S,m+n} = \mathbb{R}^3 - \{\vec{a}_1, \dots, \vec{a}_{m+n}\}$. Geometrically, the existence of splitting (17) means that Dirac's nontrivial U(1) bundle over $\mathbb{R}^3 - \{\vec{a}_1, \dots, \vec{a}_{m+n}\}$ trivializes when being embedded into an SU(2) bundle. The matrices (18) and (19) define this trivialization since $f_{NS}^{(m,n)} \mapsto \tilde{f}_{NS}^{(m,n)} = g_N^{(m,n)} f_{NS}^{(m,n)} \big(g_S^{(m,n)}\big)^{-1} = \mathbf{1}_2$.

Remark. Recall that we consider generic configurations with the conditions (2). In the case of $a_i^{1,2}$ coinciding for some $i \neq j$, one has $R^3_{N,m+n} \cup R^3_{S,m+n} \neq \mathbb{R}^3 - \{\vec{a}_1, \ldots, \vec{a}_{m+n}\}$ and the gauge potential (20) can have singularities outside $R^3_{N,m+n} \cup R^3_{S,m+n}$. For example, in the case $m = 2, n = 0$, $a_1^{1,2} = a_2^{1,2} = 0$ and $a_1^3 = -a_2^3 = a$, the gauge potential describing two separated monopoles will be singular on the interval $-a \leq x^3 \leq a$. To have nonsingular $A^{(2,0)}_{su(2)}$ one should consider $a_1^{1,2} \neq a_2^{1,2}$ or to use three open sets covering $\mathbb{R}^3 - \{\vec{a}_1, \vec{a}_2\}$ instead of two ones.

The field strength for the configuration (20) is given by

$$F^{(m,n)}_{su(2)} = \mathrm{d}A^{(m,n)}_{su(2)} + A^{(m,n)}_{su(2)} \wedge A^{(m,n)}_{su(2)} = \mathrm{i}F^{D,m+n}Q_{(m,n)} \ , \tag{21}$$

where the su(2)-valued matrix

$$Q_{(m,n)} := -\mathrm{i}g_N^{(m,n)}\sigma_3(g_N^{(m,n)})^\dagger = -\mathrm{i}g_S^{(m,n)}\sigma_3(g_S^{(m,n)})^\dagger \tag{22}$$

is well defined on $R^3_{N,m+n} \cup R^3_{S,m+n}$. It is easy to see that $Q^2_{(m,n)} = -1$ and $Q_{(m,n)}$ may be considered as the generator of the group U(1) embedded into SU(2). Then the abelian nature of the configuration (20)-(21) becomes obvious. Furthermore, for

$$A^{(m,n)}_{su(2)} = A^{(m,n)}_a \mathrm{d}x^a \quad \text{and} \quad F^{(m,n)}_{su(2)} = \frac{1}{2}F^{(m,n)}_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b \tag{23}$$

one can easily show that

$$\partial_a F^{(m,n)}_{ab} + [A^{(m,n)}_a, \, F^{(m,n)}_{ab}] = \mathrm{i}\left(\partial_a F^{D,m+n}_{ab}\right)Q_{(m,n)} \tag{24}$$

and therefore on the space $\mathbb{R}^3 - \{\vec{a}_1, \ldots, \vec{a}_{m+n}\}$ we have

$$\partial_a F^{(m,n)}_{ab} + [A^{(m,n)}_a, \, F^{(m,n)}_{ab}] = 0 \ , \tag{25}$$

which follows from the field equations describing m Dirac monopoles and n anti-monopoles. Note that the solution (20)-(23) of the SU(2) gauge theory can be embedded in any larger gauge theory following e.g. [9].

4 Point monopoles via Riemann-Hilbert problems

Here we want to rederive the described configurations by solving a matrix Riemann-Hilbert problem. For simplicity, we restrict ourselves to the case of m monopoles.

Let us consider the Bogomolny equations [10]

$$F_{ab} = \epsilon_{abc}\, D_c \chi \ , \tag{26}$$

where $D_c = \partial_c + [A_c, \cdot\ ]$ and the fields $A_a, F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ and χ take values in the Lie algebra $u(q)$. Obviously, in the abelian case $D_c\chi = \partial_c\chi$. Note that for the gauge fields $F^{D,m}$ given by (14) we have

$$F^{D,m}_{ab} = \epsilon_{abc}\, \partial_c \phi^{(m)} \quad \text{with} \quad \phi^{(m)} = \sum_{k=1}^{m} \frac{\mathrm{i}}{2r_k} \ . \tag{27}$$

Analogously, for the field $F^{(m)}_{ab}$ from (23) we have

$$F^{(m)}_{ab} = \epsilon_{abc}\, D_c \Phi^{(m)} \quad \text{with} \quad \Phi^{(m)} = \mathrm{i}\,\phi^{(m)}Q_{(m)} \ , \tag{28}$$

where $\phi^{(m)}$ is given in (27) and $Q_{(m)}$ in (22). Thus, both U(1) and SU(2) multi-monopoles as well as (m,n)-configurations (11)-(14) and (20)-(22) can be considered as solutions of the Bogomolny equations (26). In fact, the second order pure Yang-Mills equations for $F_{ab}^{D,m}$ and $F_{ab}^{(m)}$ can be obtained by differentiating (27) and (28), respectively. Moreover, in pure SU(2) Yang-Mills theory in (3+1)-dimensional Minkowski space-time, one can choose the component A_0 of the gauge potential $A{=}A_0\mathrm{d}t{+}A_a\mathrm{d}x^a$ to be nonzero and proportional to $\Phi^{(m)}$ (the abelian case is similar). Then the configuration $\{A_0^{(m)}, A_a^{(m)}\}$ will be a static multi-dyon solution of the Yang-Mills equations.

Recall that the Bogomolny equations (26) can be obtained as the compatibility conditions of the linear system

$$\big[D_{\bar y} - \frac{\lambda}{2}(D_3 + \mathrm{i}\chi)\big]\psi = 0 \quad \text{and} \quad \big[\frac{1}{2}(D_3 - \mathrm{i}\chi) + \lambda D_y\big]\psi = 0 \ , \tag{29}$$

where $D_{\bar y} = \frac{1}{2}(D_1 + \mathrm{i}D_2)$, $D_y = \frac{1}{2}(D_1 - \mathrm{i}D_2)$ and the auxiliary $q \times q$ matrix $\psi(x^a,\lambda)$ depends holomorphically on a new variable $\lambda \in U \subset \mathbb{C}P^1$. Such matrices ψ can be found via solving a parametric Riemann-Hilbert problem which is formulated in the monopole case as follows [11]. Suppose we are given a $q \times q$ matrix f_{+-} depending holomorphically on

$$\eta = y - 2\lambda x^3 - \lambda^2 \bar y \tag{30}$$

and λ for $\lambda \in U_+ \cap U_-$, where $U_+ = \mathbb{C}P^1 - \{\infty\}$ and $U_- = \mathbb{C}P^1 - \{0\}$. Then for each fixed $(x^a) \in \mathbb{R}^3$ and $\lambda \in S^1 \subset U_+ \cap U_-$ one should factorize this matrix-valued function,

$$f_{+-}(x,\lambda) = \psi_+^{-1}(x,\lambda)\psi_-(x,\lambda) \ , \tag{31}$$

in such a way that ψ_+ and ψ_- extend holomorphically in λ onto subsets of U_+ and U_-, respectively. In order to insure that $A_a^\dagger = -A_a$ and $\chi^\dagger = -\chi$ in (29) with $\psi = \psi_\pm$ one should also impose the (reality) conditions

$$f_{+-}^\dagger(x, -\bar\lambda^{-1}) \ = \ f_{+-}(x,\lambda) \quad \text{and} \quad \psi_+^\dagger(x,-\bar\lambda^{-1}) \ = \ \psi_-^{-1}(x,\lambda) \ . \tag{32}$$

After finding such $\psi_\pm$ for an educated guess of f_{+-}, one can get A_a and χ from the linear system (29) with the matrix function ψ_+ or ψ_- instead of ψ. Namely, from (29) we get

$$A_{\bar y} := \tfrac{1}{2}(A_1 + \mathrm{i}A_2) \ = \ \psi_+\partial_{\bar y}\psi_+^{-1}|_{\lambda=0} \ , \qquad A_3 - \mathrm{i}\chi \ = \ \psi_+\partial_3\psi_+^{-1}|_{\lambda=0} \ , \tag{33}$$

$$A_y := \tfrac{1}{2}(A_1 - \mathrm{i}A_2) \ = \ \psi_-\partial_y\psi_-^{-1}|_{\lambda=\infty} \ , \qquad A_3 + \mathrm{i}\chi \ = \ \psi_-\partial_3\psi_-^{-1}|_{\lambda=\infty} \ . \tag{34}$$

For more details see [11, 12] and references therein.

The construction of U(1) multi-monopole solutions via solving the Riemann-Hilbert problem for the function

$$f_{+-}^{D,m} = \frac{\lambda^m}{\prod_{k=1}^m \eta_k} =: \rho_m \quad \text{with} \quad \eta_k = \eta - h(a_k^1, a_k^2, a_k^3, \lambda) = (1{-}\lambda^2)x_k^1 + \mathrm{i}(1{+}\lambda^2)x_k^2 - 2\lambda x_k^3 \tag{35}$$

was discussed in [12] and here we describe only the SU(2) case. The ansatz for $f_{+-}^{(m)}$ which satifies (32) only for odd m was written down in the appendix C of [12]. Here we introduce the ansatz

$$f_{+-}^{(m)} = \begin{pmatrix} \rho_m & \lambda^{-m} \\ (-1)^m\lambda^m & \rho_m^{-1} + (-1)^m\rho_m^{-1} \end{pmatrix} \tag{36}$$

satisfying the reality condition (32) for any m. It is not difficult to see that

$$f_{+-}^{(m)} = \begin{pmatrix} 1 & 0 \\ (-1)^m\lambda^m\rho_m^{-1} & 1 \end{pmatrix} \begin{pmatrix} f_{+-}^{D,m} & 0 \\ 0 & (f_{+-}^{D,m})^{-1} \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-m}\rho_m^{-1} \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} f_{+-}^{D,m} & 0 \\ 0 & (f_{+-}^{D,m})^{-1} \end{pmatrix} , \quad (37)$$

where the diagonal matrix in (37) describes the Dirac line bundle L (the U(1) gauge group) embedded into the rank 2 complex vector bundle (the SU(2) gauge group) as $L \oplus L^{-1}$. This gives another proof of the equivalence of U(1) and SU(2) point monopole configurations (see [12] for more details). Furthermore, the matrix (36) can be splitted as follows:

$$f_{+-}^{(m)} = \big(\psi_+^{(m)}\big)^{-1}\psi_-^{(m)} , \quad (38)$$

where

$$\psi_+^{(m)} = \hat{\psi}_+^{(m)} \begin{pmatrix} 1 & 0 \\ (-1)^{m+1}\lambda^m\rho_m^{-1} & 1 \end{pmatrix} , \quad \psi_-^{(m)} = \hat{\psi}_-^{(m)} \begin{pmatrix} 1 & -\lambda^{-m}\rho_m^{-1} \\ 0 & 1 \end{pmatrix} , \quad (39)$$

$$\hat{\psi}_+^{(m)} = g_S^{(m)} \begin{pmatrix} \psi_+^{S,m} & 0 \\ 0 & (\psi_+^{S,m})^{-1} \end{pmatrix} , \quad \hat{\psi}_-^{(m)} = g_N^{(m)} \begin{pmatrix} \psi_-^{N,m} & 0 \\ 0 & (\psi_-^{N,m})^{-1} \end{pmatrix} , \quad (40)$$

$$\psi_+^{S,m} = \prod_{i=1}^m \psi_+^S(x_i^a,\lambda) , \quad \psi_+^S(x_i^a,\lambda) = \xi_+(x_i^a) - \lambda\xi_+^{-1}(x_i^a)\bar{y}_i , \quad \xi_+(x_i^a) = (r_i - x_i^3)^{\frac{1}{2}} , \quad (41)$$

$$\psi_-^{N,m} = \prod_{i=1}^m \psi_-^N(x_i^a,\lambda) , \quad \big(\psi_-^N(x_i^a,\lambda)\big)^{-1} = \xi_-(x_i^a)\bar{y}_i + \lambda^{-1}\xi_-^{-1}(x_i^a) , \quad \xi_-(x_i^a) = (r_i + x_i^3)^{-\frac{1}{2}} . \quad (42)$$

The explicit form of $g_N^{(m)}$ and $g_S^{(m)}$ is given in (18) and (19). Note that both $\psi_\pm^{(m)}$ and $\hat{\psi}_\pm^{(m)}$ satisfy the reality conditions (32).

Formulae (38)-(42) solve the parametric Riemann-Hilbert problem for our $f_{+-}^{(m)}$ restricted to a contour on $\mathbb{C}P^1$ which avoids all zeros of the function $\prod_{k=1}^m \eta_k$. Substituting (39)-(42) into formulae (33)-(34), we get

$$A_{\bar{y}}^{(m)} = \hat{g}_S^{(m)}\partial_{\bar{y}}\big(\hat{g}_S^{(m)}\big)^{-1}, \; A_y^{(m)} = \hat{g}_N^{(m)}\partial_y\big(\hat{g}_N^{(m)}\big)^{-1}, \; A_3^{(m)} = g_S^{(m)}\partial_3\big(g_S^{(m)}\big)^\dagger = g_N^{(m)}\partial_3\big(g_N^{(m)}\big)^\dagger, \quad (43)$$

$$\chi^{(m)} = \frac{\mathrm{i}}{2}\Big(\hat{g}_S^{(m)}\partial_3\big(\hat{g}_S^{(m)}\big)^{-1} - \hat{g}_N^{(m)}\partial_3\big(\hat{g}_N^{(m)}\big)^{-1}\Big) , \quad (44)$$

where

$$\hat{g}_S^{(m)} = g_S^{(m)} \begin{pmatrix} \xi_+ & 0 \\ 0 & \xi_+^{-1} \end{pmatrix} \text{ with } \; g_S^{(m)} = \frac{1}{\Big(\prod_{i=1}^m (r_i - x_i^3)^2 + \prod_{i=1}^m y_i\bar{y}_i\Big)^{\frac{1}{2}}} \begin{pmatrix} \prod_{j=1}^m (r_j - x_j^3) & \prod_{j=1}^m \bar{y}_j \\ -\prod_{j=1}^m y_j & \prod_{j=1}^m (r_j - x_j^3) \end{pmatrix} , \quad (45)$$

$$\hat{g}_N^{(m)} = g_N^{(m)} \begin{pmatrix} \xi_-^{-1} & 0 \\ 0 & \xi_- \end{pmatrix} \text{ with } \; g_N^{(m)} = \frac{1}{\Big(\prod_{i=1}^m (r_i + x_i^3)^2 + \prod_{i=1}^m y_i\bar{y}_i\Big)^{\frac{1}{2}}} \begin{pmatrix} \prod_{j=1}^m \bar{y}_j & \prod_{j=1}^m (r_j + x_j^3) \\ -\prod_{j=1}^m (r_j + x_j^3) & \prod_{j=1}^m y_j \end{pmatrix} \quad (46)$$

and

$$\xi_+ = \prod_{k=1}^{m} \xi_+(x_k^a) = \prod_{k=1}^{m} (r_k - x_k^3)^{\frac{1}{2}} \ , \quad \xi_- = \prod_{k=1}^{m} \xi_-(x_k^a) = \prod_{k=1}^{m} (r_k + x_k^3)^{-\frac{1}{2}} \ . \tag{47}$$

It is not difficult to see that the configuration (43) coincides with (20) and $\chi^{(m)}$ from (44) with $\Phi^{(m)}$ from (28). Thus, we have derived SU(2) multi-monopole point-like solutions via a parametric Riemann-Hilbert problem.

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